# Decay of Coherent Scalar Field Oscillations: Two Approaches

By

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Submitted in partial fulfillment
of the requirements for:

Undergraduate Departmental Honors
Department of Physics and Astronomy, University of New Mexico

May, 2020

## Acknowledgements

*I would like to extend a heartfelt acknowledgement to Prof. Rouzbeh Allahverdi, my advisor, for his years of mentorship, advice, conversation, and coffee. His guidance and generosity throughout my time at UNM has been profoundly impactful, both in deepening my understanding of physics, and in enabling me to begin pursuing and achieving my academic and career goals.*

*I would also like to acknowledge the partial support for this project from the Rayburn Research Fund, and NSF Grant No. PHY-1720174*

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**Abstract**

The decay of coherent scalar field oscillations is thought to be an important mechanism for standard model particle production in the early universe. Because of this, it is important to understand where different computational approaches for studying decay agree and disagree. We show that for two simple theories involving coherent oscillations, when the time-scale of the energy transfer is much larger than the time-scale of the coherent oscillations, the scalar field may be approximated as a collection of quanta, and standard S-matrix techniques become applicable. We show this explicitly to first order by comparing the standard particle-decay results with those obtained from a classical background approach. We argue that for sufficiently weak interactions, and sufficiently slow energy transfer rates, equality holds at all orders. Further, we argue that when interactions are strong, the energy transfer rate can become fast, and the coherent oscillations must be treated using classical background methods.

# Contents

# 1 Introduction

## 1.1 Background

Scalar fields play an important role in many areas of modern physics. Such spin-0 fields are used to describe the Higg's field in the standard model of particle physics, they appear as the super-partners of quarks and leptons in supersymmetric extensions of the standard model, and they arise in the context of string theory as moduli which determine the geometry of extra spatial dimensions. Here, we take our primary motivation from coherently oscillating scalar fields which arise in the context of early-universe cosmology. By coherent oscillations we simply refer to oscillations of wavenumber zero; As a simple example, this can occur when electromagnetic waves oscillate at plasma frequency.

When scalar fields are displaced during inflation (for example, the inflaton), as the universe expands very rapidly, the length scale of the displacements becomes extremely large. When this happens, the displacements can be regarded as approximately uniform over large spatial regions (i.e. coherence). This can result in coherent oscillations which interact with the standard model fields and decay rapidly into standard model particles. An important example is the decay of the inflaton field, which is thought to be responsible for the creation of matter and anti-matter in the universe.

## 1.2 Computational Approaches

Standard computational approaches for analyzing quantum field decay often rely on the use of asymptotic particle states, and study their interactions via the S-matrix [1]. This approach requires that energy transfer processes occur very slowly with respect to the relevant time scales of the system. Such techniques may be inadequate for studying the decay of coherent scalar field oscillations, since the rate of energy transfer can become fast relative to the frequency of field oscillations, due to the coherence of the field.

In order to treat the problem more accurately, one desires an approach which utilizes the time-dependent Hamiltonian interaction between the oscillating scalar field and the Standard Model fields. The elementary treatment is one in which the oscillating scalar field is regarded as a classical background field which induces time-dependent Hamiltonian interactions. As an example, for a coherently oscillating field $\phi$ of mass $m_\phi$, the oscillations

$$\phi(t) = \phi_0 \cos(m_\phi t) \tag{1}$$

can induce interaction terms in the Hamiltonian of the form

$$g\phi^2(t)\chi^2, \quad f\phi(t)\chi^3, \quad \sigma\phi(t)\chi\psi\overline{\psi}, \quad \cdots, \quad \text{etc.} \tag{2}$$

where $\chi$ and $\psi$ denote some standard model spin-0 bosons or spin-1/2 fermions, respectively, and where the coupling constants $g$, $f$, $\sigma$, etc. determine the strength of the interactions. In each case, we can regard the products $g\phi^2(t)$, $f\phi(t)$, $\sigma\phi(t)$, etc. as constituting time-dependent couplings.

In such cases, to study the rate of particle production we calculate the vacuum excitation amplitudes using the propagator induced by the time-dependent Hamiltonian interaction $H_{int}(t)$. For example, with $H_{int}(t) = g\phi(t)\chi^3$, we may be interested in the vacuum excitation amplitude

$$\lim_{T\to\infty} \langle\chi\chi\chi|U(T,-T)|0\rangle = \lim_{T\to\infty} \langle\chi\chi\chi| \exp\left\{\int_{-T}^{T} dt \int d^3\mathbf{x}\, H_{int}(t)\right\}|0\rangle \tag{3}$$

However, when the rate of energy transfer $\Gamma$ is sufficiently slow, it is reasonable to expect that we may work in the Born-Oppenheimer approximation [2] with $\Gamma << m_\phi$, and average over the fast $\phi(t)$ oscillations to obtain a scalar field $\phi$ which behaves like a collection of quanta, so that its interaction with SM fields can be modeled approximately using standard S-matrix techniques. It has been shown, however, that for some simple interactions like

$$\mathcal{L}_{int} = -f\phi\overline{\psi}\psi - (g\phi + h\phi^2)\chi^2 \tag{4}$$

this 'particle picture' is not adequate for computing the rate of $\psi$ and $\chi$ production [3]. The reason for this is twofold: For quadratic interactions of this nature, the particle picture does *not* account for the phenomenon of parametric resonance, which yields higher production rates for particles near the mass of $\phi$. Furthermore, in the case of broad parametric resonance

[4], production of $\chi$ particles becomes very rapid, occurring on time scales much smaller than the time-scale of $\phi$ oscillations. As a consequence, the approximation $\Gamma << m_\phi$ is violated, and the standard particle approach is insufficient.

In what follows, we investigate the adequacy of the standard particle approach in different physical limits for some simple theories. In chapter two we consider the interaction $H_{int} = g\phi\chi^3$, and compare the classical background computational approach with the S-matrix 'particle approach' in the limit of fast oscillation and small coupling. In chapter three we study these two approaches for a fermionic interaction $H_{int} = \sigma\phi\chi\overline{\psi}\psi$ in the same limits. Finally, we conclude in chapter 4 by discussion higher order processes, and by determining the physical limits in which the particle approximation becomes inadequate.

## 2 The $\phi\chi^3$ Interaction

We begin by considering the system of interacting scalar fields $\phi$ and $\chi$ described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m_\phi^2\phi^2 + \frac{1}{2}(\partial\chi)^2 - \frac{1}{2}m_\chi\chi^2 - g\phi\chi^3 + \cdots \tag{5}$$

with interaction terms $\mathcal{L}_{int} = -g\phi\chi^3 + \cdots$. In general, we must include higher order interactions, e.g a $\chi^4$ term, to ensure that the interaction potential is bounded below, and hence ensure the existence of a ground state. This will not affect the results our of present calculations.

We begin this section by studying the interaction $\mathcal{L}_{int}$ from the particle perspective, whereby we will work in the Born-Oppenheimer approximation, i.e. the limit of fast oscillations ($\Gamma << m_\phi$), and regard $\phi$ as a quantum field amenable to a standard S-matrix treatment. In particular, we will calculate the decay rate for the first order process $\phi \to \chi + \chi + \chi$. Next, we will use the classical background approach, regarding $\phi$ as a fully time-dependent field $\phi = \phi(t)$, and calculate the amplitude for vacuum excitation to a state of three $\chi$ quanta. We conclude by showing that the classical background results reduce to the particle approach results in the limit $m_\phi >> \Gamma$.

## 2.1 The Particle Approximation

Here, we assume that $\phi$ behaves as a collection of quanta of mass $m_\phi$, at rest. These quanta are constructed, for example, from averaging over the fast oscillations of a scalar field $\phi = \phi_0 \cos(m_\phi t)$. We wish to find the rate of particle production in the limit of small coupling $g$ through the first order process depicted in figure 1.

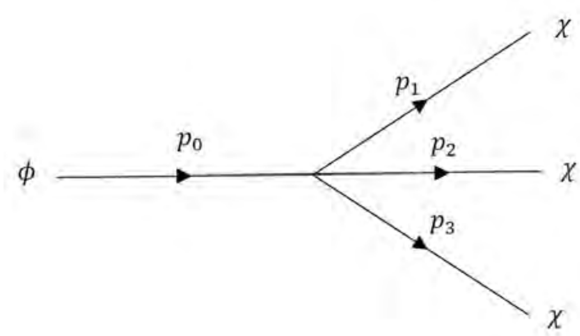


Figure 1: First order process in $\phi\chi^4$ interaction.

We find from Fermi's golden rule [1] that in the rest frame of the $\phi$ field, the differential decay rate is given by

$$d\Gamma = \frac{1}{2m_\phi}\left(\prod_{i=1}^{3}\frac{d^3p_i}{(2\pi)^3 2E_i}\right)|\mathcal{M}|^2(2\pi)^4\delta^{(4)}(p_0 - \sum_{i=1}^{3}p_i) \tag{6}$$

where $\mathcal{M}$ denotes the amplitude for the first order process. For this simple process we have that $\mathcal{M} = ig$. After integrating over the angular components of the particle phase space (see appendix A.1 for details) we find that the differential decay rate $d\Gamma$ can be written in Dalitz coordiates [5] as

$$d\Gamma = \frac{|\mathcal{M}|^2}{32m_\phi^3(2\pi)^3}dm_{12}^2 dm_{13}^2 = \frac{g}{32m_\phi^3(2\pi)^3}dm_{12}^2 dm_{13}^2 \tag{7}$$

with $m_{ij}$ defined by

$$m_{ij}^2 \equiv (p_i + p_j)^2 \tag{8}$$

The region of integration, known as the Dalitz plot, is given in figure 2 below:

In general,there is no closed form for $\Gamma_\chi$, but in the limit of massless identical $\chi$ particles,

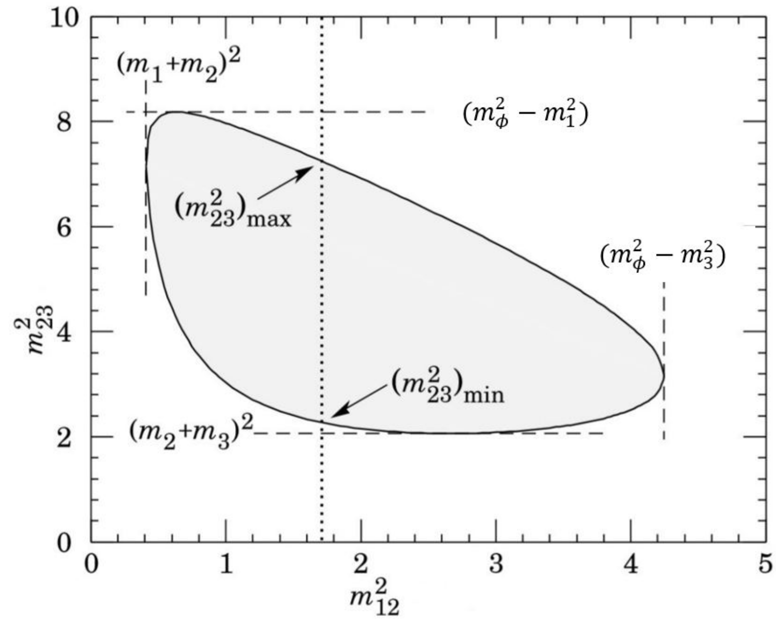


Figure 2: Dalitz plot. Produced from [5]

i.e. $m_\chi = 0$, the Dalitz plot reduces to the lower-triangular region described by

$$m^2_{13-max} = m^2_\phi - m^2_{12} \tag{9}$$

$$m^2_{13-min} = 0 \tag{10}$$

In this case, performing the integral yields the final result

$$\Gamma_\chi = \frac{3g^2 m_\phi}{32(2\pi)^3}, \qquad (m_\chi = 0) \tag{11}$$

where we have included an extra factor 6 due to symmetry in the case of identical mass-less particles.

## 2.2 The Classical Background Approach

We now consider the system described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m_\phi^2\phi^2 + \frac{1}{2}(\partial\chi)^2 - \frac{1}{2}m_\chi\chi^2 - g\phi(t)\chi^3 + \cdots \tag{12}$$

where $\phi(t) = \phi_0 \cos(m_\phi t)$ describes a coherently oscillating scalar field. As before, we must include a $\chi^4$ term to ensure the existence of a ground state, but this won't affect our calculations.

We are interested in finding the amplitude for vacuum excitation in the form of three $\chi$ particles in the limit of small coupling $g$. We will assume that the time scale of the energy transfer is $\Delta T = 2T$, beginning at time $-T$ and completing at time $T$. That is, we wish to compute the amplitude

$$\mathcal{M} = \langle \mathbf{p}_1 \mathbf{p}_2 \mathbf{p}_3 | \exp \left\{ \int_{-T}^{T} dt \int d^3\mathbf{x} \, H_{int}(t) \right\} |0\rangle \tag{13}$$

$$= \langle \mathbf{p}_1 \mathbf{p}_2 \mathbf{p}_3 | \int_{-T}^{T} dt \int d^3\mathbf{x} \, g\phi(t)\chi^3(x) |0\rangle + \cdots \tag{14}$$

$$= ig\phi_0 \int_{-T}^{T} dt \int d^3\mathbf{x} \langle \mathbf{p_1 p_2 p_3} | \cos(m_\phi t)\chi^3(x)|0\rangle + \cdots \tag{15}$$

We find (see appendix B.1 for details) that upon expanding the field $\chi(x)$, performing the integration over the time interval $\Delta T$, and utilizing a few computational tricks, that the square of the amplitude is given by

$$|\mathcal{M}|^2 = g^2\phi_0^2 \Bigg[ \frac{\sin^2\left((E_1+E_2+E_3-m_\phi)T\right)}{(E_1+E_2+E_3-m_\phi)^2} + \frac{\sin^2\left((E_1+E_2+E_3+m_\phi)T\right)}{(E_1+E_2+E_3+m_\phi)^2} + 2\frac{\sin\left((E_1+E_2+E_3-m_\phi)T\right)\sin\left((E_1+E_2+E_3+m_\phi)T\right)}{(E_1+E_2+E_3)^2-m_\phi^2} \Bigg] \times V \times (2\pi)^3\delta^{(3)}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) \tag{16}$$

where $V$ denotes the volume of phase space being integrated over. Let us denote the quantity in brackets by $\Big[\cdots\Big] =: \mathcal{S}$, and each of its terms by $\mathcal{S}_-$, $\mathcal{S}_+$, and $\mathcal{S}_\pm$, respectively, to emphasize the differing signs carried by $m_\phi$ in the respective arguments of the sine functions.

Using our expression for the squared amplitude $|\mathcal{M}|^2$, the differential probability for the process to occur in the volume of momentum phase space centered at $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ is:

$$dP = \left( \prod_{i=1}^{3} \frac{d^3p_i}{(2\pi)^3 2E_i} \right) g^2\phi_0^2 \times \mathcal{S} \times V \times (2\pi)^3\delta^{(3)}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) \tag{17}$$

Now, taking note of the field density equation $\rho = m_\phi^2 \phi_0^2$ we obtain the number density

$$n_\phi = \frac{N}{V} = \frac{\rho}{m_\phi} = m_\phi \phi_0^2 \tag{18}$$

With $N \equiv 1$ for decay of a single $\phi(t)$ quanta, we can use $\phi_0^2 = \frac{1}{m_\phi V}$ to write the propbability as

$$dP = \frac{1}{m_\phi} \left( \prod_{i=1}^{3} \frac{d^3 p_i}{(2\pi)^3 2E_i} \right) g^2 \times \mathcal{S} \times (2\pi)^3 \delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) \tag{19}$$

Or, since the interaction occurs over a time period $\Delta T$, the differential excitation rate is

$$d\Gamma = \frac{dP}{\Delta T} = \frac{1}{\Delta T m_\phi} \left( \prod_{i=1}^{3} \frac{d^3 p_i}{(2\pi)^3 2E_i} \right) g^2 \times \mathcal{S} \times (2\pi)^3 \delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) \tag{20}$$

We wish to show that in the limit as the time-scale of the energy transfer $\Delta T$ is large with respect to the time-scale of $\phi$ oscillations, i.e. $\Delta T >> 1/m_\phi$, $\Gamma$ reduces to the expression found in the S-matrix approach. Recalling the results from section 2.1, we see that the above expression agrees with the 'particle picture' differential decay rate (6) (with $|\mathcal{M}|^2 = g^2$) if and only if $\mathcal{S}$ reduces to the delta function $(\pi \Delta T)\delta(E_1 + E_2 + E_3 - m_\phi)$. We will show that this is precisely the case when $\Delta T >> 1/m_\phi$. First, consider the leading term $\mathcal{S}_-$. We may rewrite this as

$$\mathcal{S}_- = \left( \frac{\Delta T}{2} \right)^2 \frac{\sin^2 [(E_1 + E_2 + E_3 - m_\phi)T]}{[(E_1 + E_2 + E_3 - m_\phi)T]^2} \tag{21}$$

where $\Delta T = 2T$ is the duration of the interaction. Observe that this behaves like the function $\sin^2(x)/x^2$ with $x = (E_1 + E_2 + E_3 - m_\phi)T$. A plot of this function is given in figure 3 below:

We see that $\mathcal{S}_-$ as a function of energy is peaked near $E_1 + E_2 + E_3 = m_\phi$ with amplitude $(\Delta T/2)^2$ and width $\sim 2\pi/T = 4\pi/\Delta T$. Thus, in the limit as $\Delta T >> \frac{1}{m_\phi}$, the amplitude of the peak at $E_1 + E_2 + E_3 = m_\phi$ approaches infinity, and the width of the peak vanishes. Moreover, the area under $\mathcal{S}_-$ scales like $(\frac{\Delta T}{2})^2 \times \frac{4\pi}{\Delta T} = \pi \Delta T$, and hence $S_-$ approaches a delta function $(\pi \Delta T)\delta(E_1 + E_2 + E_3 - m_\phi)$.

Now consider the second term $\mathcal{S}_+$. A similar result holds: This term approaches the delta function $(\pi \Delta T)\delta(E_1 + E_2 + E_3 + m_\phi)$. Now, of course, $E_i > 0$ and $m_\phi > 0$, so that this delta function vanishes under the integral. Finally, consider the third sinusoidal term. Each

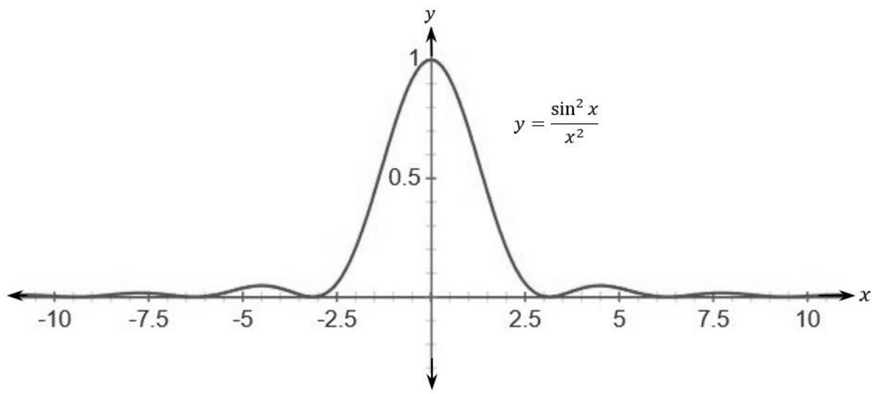


Figure 3: Plot of $\sin^2(x)/x^2$.

of the two sinusoids in $\mathcal{S}_\pm$ is peaked at height $\sim \Delta T$ when their respective arguments are near zero. However, each factor is significantly suppressed by the other, since their peaks peaks do not overlap. Thus, even near $E_1 + E_2 + E_3 = m_\phi$, this term is negligible in the $\Delta T >> 1/m_\phi$ limit.

Hence, in the $\Delta T >> 1/m_\phi$ limit, $\mathcal{S} = \mathcal{S}_- + \mathcal{S}_+ + \mathcal{S}_\pm$ reduces to the desired delta-function $(\pi \Delta T)\delta(E_1 + E_2 + E_3 - m_\phi)$, and hence the expression for the differential decay rate becomes

$$d\Gamma = \frac{1}{2m_\phi}\left(\prod_{i=1}^{3} \frac{d^3 p_i}{(2\pi)^3 2E_i}\right) \times g^2 \times (2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3 - p_0), \qquad \Delta T >> \frac{1}{m_\phi} \tag{22}$$

Comparing with equation (6), we see that with $\Delta T >> m_\phi$,

$$\Gamma_{background} = \Gamma_{particle} \tag{23}$$

As a final important remark, recall from equation (11) that in the particle picture we found $\Gamma \propto g^2 m_\phi$. The limit of small coupling $g^2 << 1$ in the particle picture is therfore consistent with the Born-Oppenheimer approximation (wherein the time-scale of the energy transfer $\Delta T \sim 1/\Gamma$ is much less than time-scale of oscillations, i.e. $\Gamma << m_\phi$) utilized in averaging over fast $\phi(t)$ oscillations. This provides further evidence that for small coupling, the condition $\Gamma << m_\phi$ is sufficient to ensure that the particle approximation is adequate.

# 3 The $\phi\chi\overline{\psi}\psi$ Interaction

Having shown the agreement between the particle picture and the classical background picture in the Born-Oppenheimer approximation and small coupling limit for the $\phi\chi^3$ theory, we now turn to the fermionic interaction in the system described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m_\phi^2\phi^2 + \frac{1}{2}(\partial\chi)^2 - \frac{1}{2}m_\chi\chi^2 - \sigma\phi\chi\overline{\psi}\psi + \cdots \tag{24}$$

where $\phi$ and $\chi$ are again two real scalar fields, and $\psi$ is a spin-$\frac{1}{2}$ field. We begin by working in the particle picture to calculate the decay rate for the first-order process $\phi \to \chi + \overline{\psi} + \psi$ in the interaction $\mathcal{L}_{int} = -\sigma\phi\chi\overline{\psi}\psi$. Next, we use the classical background approach to calculate the rate of vacuum excitation to a state consisting of one $\chi$ quanta, and a fermion-antifermion pair $\overline{\psi} + \psi$. We conclude by showing that the two results agree in the Born-Oppenheimer approximation.

## 3.1 The Particle Approximation

We again begin by supposing that we are working in the limit where the rate of energy transfer is must slower than the rate of oscillations in the coherently oscillating field $\phi(t)$, allowing us to generate the scalar field $\phi$ by averaging over the rapid oscillations. We then proceed with the standard particle-interaction picture from standard quantum field theory. The first order $\phi \longrightarrow \chi + \bar{\psi} + \psi$ decay process is characterized by the Feynman diagram in figure 4.

We find (see A.2) that after summing over the final spin states $r$ and $s$ of $\overline{\psi}$ and $\psi$, that the square of the 1st order amplitude is

$$\sum_{r,s} |\mathcal{M}|^2 = 4\sigma^2(p_2 \cdot p_3 - m_2 m_3) \tag{25}$$

Fermi's golden rule give's the differential decay rate in the $\phi$ rest frame as

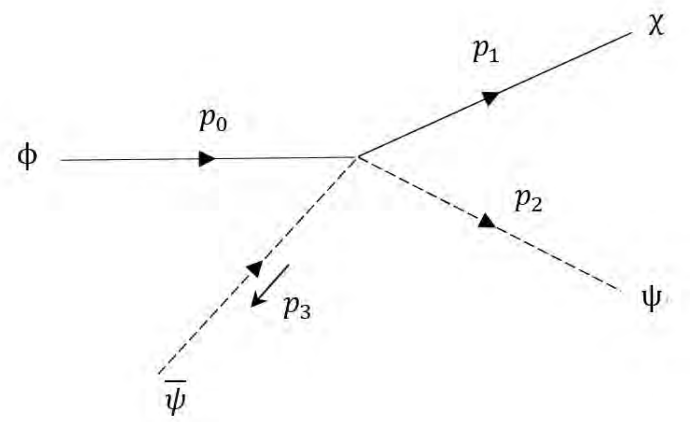


Figure 4: First order process in $\phi\chi\overline{\psi}\psi$ interaction

$$d\Gamma = \frac{1}{2m_\phi}\left(\prod_{i=1}^{3}\frac{d^3p_i}{(2\pi)^3 2E_i}\right)\sum_{r,s}|\mathcal{M}|^2(2\pi)^4\delta^{(4)}(p_0 - \sum_{i=1}^{3}p_i) \tag{26}$$

$$= \frac{1}{2m_\phi}\left(\prod_{i=1}^{3}\frac{d^3p_i}{(2\pi)^3 2E_i}\right)4\sigma^2(p_2\cdot p_3 - m_2 m_3)(2\pi)^4\delta^{(4)}(p_0 - \sum_{i=1}^{3}p_i) \tag{27}$$

After simplifying the expression for the differential decay rate, we find that

$$d\Gamma = \sigma^2\frac{(E_2+E_3)^2 - E_1^2 - (m_2+m_3)^2 + m_1^2}{4m_0(2\pi)^3}dE_2 dE_3 \tag{28}$$

Working again in the Dalitz coordinates defined by $m_{ij} \equiv (p_i + p_j)^2$, we find that this integral becomes

$$d\Gamma = \sigma^2\frac{m_0^2 + m_1^2 - 2m_2m_3 - (m_{12}^2 + m_{13}^2)}{16m_0^3(2\pi)^3}dm_{12}^2 dm_{13}^2 \tag{29}$$

where the domain of integration is again as depicted in figure 2.

Again, the general integral does not have a closed form, but we may reduce to the special case of massless products, i.e. $m_\chi = m_\psi = 0$, at which point the Dalitz plot again simplifies to the triangular region described by $m_{13-max}^2 = m_0^2 - m_{12}^2$, $m_{13-min}^2 = 0$. In this case, the differential decay rate becomes

$$d\Gamma = \sigma^2 \frac{m_\phi^2 - (m_{12}^2 + m_{13}^2)}{16 m_\phi^3 (2\pi)^3} dm_{12}^2 dm_{13}^2 \tag{30}$$

and performing the integration yields

$$\Gamma = \sigma^2 \left( \frac{m_\phi^2}{16 m_\phi^3 (2\pi)^3} \frac{m_\phi^4}{2} - \frac{1}{16 m_\phi^3 (2\pi)^3} \frac{m_\phi^6}{3} \right) = \frac{\sigma^2 m_\phi^3}{96 (2\pi)^3} \tag{31}$$

## 3.2 The Classical Background Approach

We now turn to the interaction

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2} m_\phi^2 \phi^2 + \frac{1}{2}(\partial\chi)^2 - \frac{1}{2} m_\chi \chi^2 - \sigma\phi(t)\chi\overline{\psi}\psi + \cdots \tag{32}$$

where $\phi(t) = \phi_0 \cos(m_\phi t)$ is treated as a classical background. As was the case for the $g\phi(t)\chi^3$ interaction we wish to calculate the vacuum excitation amplitude

$$\begin{aligned}
\mathcal{M} &= \langle \mathbf{p}_1 \mathbf{p}_2 \mathbf{p}_3 | \exp\left\{ \int_{-T}^{T} dt \int d^3\mathbf{x}\, H_{int}(t) \right\} |0\rangle \\
&= \langle \mathbf{p}_1 \mathbf{p}_2 \mathbf{p}_3 | \int_{-T}^{T} dt \int d^3\mathbf{x}\, \sigma\phi(t)\chi(x)\overline{\psi}(x)\psi(x) |0\rangle + \cdots \\
&= i\sigma\phi_0 \int_{-T}^{T} dt \int d^3\mathbf{x} \langle \mathbf{p_1 p_2 p_3} | \cos(m_\phi t)\chi(x)\overline{\psi}(x)\psi(x)|0\rangle + \cdots
\end{aligned} \tag{33}$$

We find upon computing the first order amplitude (details are contained in appendix B.2), that the spin-average square of the amplitude is given by

$$\begin{aligned}
\sum_{r,s} |\mathcal{M}|^2 = 4\sigma^2\phi_0^2 (p_2 \cdot p_3 - m_2 m_3) \Bigg[ & \frac{\sin^2\left((E_1+E_2+E_3-m_\phi)T\right)}{(E_1+E_2+E_3-m_\phi)^2} \\
& + \frac{\sin^2\left((E_1+E_2+E_3+m_\phi)T\right)}{(E_1+E_2+E_3+m_\phi)^2} \\
+ 2\frac{\sin\left((E_1+E_2+E_3-m_\phi)T\right)\sin\left((E_1+E_2+E_3+m_\phi)T\right)}{(E_1+E_2+E_3)^2 - m_\phi^2} \Bigg] & \qquad (34) \\
\times V \times (2\pi)^3 \delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) & \qquad (35)
\end{aligned}$$

where $V$ is again the volume of the particle phase space, and where the bracketed quantity is *precisely* the quantity $\mathcal{S} := \mathcal{S}_- + \mathcal{S}_+ + \mathcal{S}_\pm$ found for the $g\phi\chi^3$ interaction in section 2.2. Because the same factor $\mathcal{S}$ appears here, our analysis of the $\Delta T >> 1/m_\phi$ limit in section 2.1 applies here, wholesale. Utilizing again the number density (18) to replace $\phi_0$, we find that the differential decay rate $d\Gamma = dP/\Delta T$ obtained from (34) is

$$\begin{aligned} d\Gamma &= \frac{1}{2}\left(\prod_{i=1}^{3}\frac{d^3p_i}{(2\pi)^3 2E_i}\right)4\sigma^2\phi_0^2(p_2\cdot p_3 - m_2m_3)\times\mathcal{S}\times V\times(2\pi)^3\delta^{(3)}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) \\ &= \frac{1}{2m_\phi}\left(\prod_{i=1}^{3}\frac{d^3p_i}{(2\pi)^3 2E_i}\right)4\sigma^2(p_2\cdot p_3 - m_2m_3)|^2(2\pi)^4\delta^{(4)}(p_0 - \sum_{i=1}^{3}p_i) \end{aligned} \tag{36}$$

which agrees with the differential decay rate (26) from the particle approach with $\sum_{r,s}|\mathcal{M}|^2 = 4\sigma^2(p_2\cdot p_3 - m_2m_3)$.

Therefore, we see again that

$$\Gamma_{background} = \Gamma_{particle} \tag{37}$$

in the $\Delta T >> 1/m_\phi$ limit. Now, notice that from our result (31) for the particle-decay rate, $\Gamma \propto \sigma^2 m_\phi^3 = (\sigma m_\phi)^2 m_\phi$, so that the Born-approximation $\Gamma << m_\phi$ is achieved when $\sigma m_\phi << 1$. This tells us, again, that for small coupling $\sigma << 1/m_\phi$, the condition $\Gamma << m_\phi$ ensures the particle description is adequate.

# 4 Discussion

## 4.1 Higher Order Processes

We have seen explicitly that for the $g\phi\chi^3$ and $\sigma\phi\chi\overline{\psi}\psi$ interactions, a particle treatment is adequate for calculating the first order decay rate of coherent $\phi(t)$ oscillations in the limit that the energy transfer rate $\Gamma$ is small compared to the frequency of $\phi(t)$ oscillations ($\Gamma << m_\phi$). We established this result in detail for first order processes. However, we argue that such agreement is seen even at higher orders.

The results of the time-dependent treatment in appendices B.1 and B.2 hint at what we can expect from higher order calculations. Our ability to make connection with the particle approach relies on the fact that upon integrating over $\phi(t)$, we obtain sinusoidal terms which approach energy delta functions in the $\Delta T >> 1/m_\phi$ limit. We can expect that this fact holds true to all orders, for sufficiently small $g$ and $\Gamma$. Indeed, consider the nth order term in the the Dyson series expansion of the propagator:

$$U(T,-T) = 1 + (-i)\int_{-T}^{T} dt_1 H_{int}(t_1) + \int_{-T}^{T} dt_2 \mathcal{T}\{H_{int}(t_1)H_{int}(t_2)\} + \cdots$$

where $\mathcal{T}$ denotes the time-ordering operation. Since the oscillations $\phi(t)$ are coherent, the time-dependence resulting from $\phi$ is contained in a factor of the form

$$\cos(m_\phi t) \times \cdots \times \cos(m_\phi t_n)$$

which we may write in terms of complex exponents. When we perform the n-fold time integrals, we are still able to isolate the time-dependence of the amplitude in a collection of terms (like the bracketed expression denoted by $\mathcal{S}$, encountered in 2.2 and 3.2) containing sinusoids of the form

$$\frac{\sin^n\left(E_1 + E_2 + E_3 \pm m_\phi)T\right)}{(E_1 + E_2 + E_3 \pm m_\phi)^n},$$

along with cross-terms (like $\mathcal{S}_\pm$ in aforementioned sections).

It is still the case, as it was for $n = 1$, that these are peaked near $E_1 + E_2 + E_3 = \pm m_\phi$, and approach energy delta functions in the limit that duration of the interaction is large: $\Delta T >> 1/m_\phi$. But since $E_i \geq 0$, $m_\phi \geq 0$, the only term which survives in the $T >> 1/m_\phi$ limit is the single delta function $\delta(E_1 + E_2 + E_3 - m_\phi)$. Thus, we might expect the particle approach and classical background approaches to agree to arbitrarily high order when the coupling is sufficiently small, and $\Gamma$ is sufficiently small.

## 4.2 Inadequacies of Particle Approach

Despite the expected agreement to all orders (for sufficiently small coupling, and sufficiently fast oscillations), we expect that for strong interactions, the single-particle decay model does not adequately describe the decay of $\phi(t)$. For strong interactions (e.g. large $g\phi_0$ in the $g\phi(t)\chi^3$ interation), the coherence of the scalar field becomes important, and we can no longer expect it to decay as single quanta. We expect higher order processes to become important (e.g. two $\phi$ quanta producing 4 $\chi$ quanta). As higher order processes contribute more to the decay rate of the field, we expect that after summing over all orders, the total decay rate becomes large, and the Born-Oppenheimer approximation $\Gamma << m_\phi$ breaks down.

We can find estimates on the strength $\phi_0$ and number density $n_\phi$ at which the Born-Oppenheimer approximations is violated. Consider first the $g\phi\chi^3$ interaction in the classical background picture. We know that the amplitude for second order processes satisfies $\mathcal{M}^{(2)} \propto g^2\phi_0^2$. Hence, we expect that $|\mathcal{M}^{(2)}|^2 \propto g^4\phi_0^4$ so that $\Gamma^{(2)}$,which carries dimensions of mass (in natural units), must scale as

$$\Gamma^{(2)} \sim g^4\phi_0^4/m_\phi^3$$

A quick note: Here, I am using $\sim$ to denote proportionality by a dimensionless constant, rather than to denote proportionality by a constant of order 1. We know, moreover, that the first order amplitude in the classical background approach must be proportional to $g^2\phi_0^2$, and hence scales like $\Gamma^{(1)} \sim g^2\phi_0^2/m_\phi$. Therefore, the second and first order processes contribute comparably to $\Gamma$ when

$$\frac{\Gamma^{(2)}}{\Gamma^{(1)}} \sim \frac{g^2\phi_0^2}{m_\phi^2} \gtrsim 1$$

(ignoring all numerical factors, which might not be of order one). Thus, with $n_\phi = m_\phi\phi_0^2$, we see that higher order processes become significant when

$$\phi_0^2 >> m_\phi^2/g^2 \qquad \Longleftrightarrow \qquad n_\phi >> m_\phi^3/g^2 \tag{38}$$

and in this limit, the Born-Oppenheimer approximation $\Gamma << m_\phi$ is violated. Note, that this is again corroborated by the fact that for $\Gamma^{(1)} \sim g^2\phi_0^2/m_\phi$, the condition $\Gamma^{(1)} >> m_\phi$ is precisely (38)

Now, consider the $\sigma\phi\chi\overline{\psi}\psi$ interaction. Again, we have that in the classical background picture, to first order, $\Gamma^{(1)} \propto \sigma^2\phi_0^2$, so that, carrying dimensions of mass, $\Gamma^{(1)}$ must scale as

$$\Gamma^{(1)} \sim \sigma^2\phi_0^2 m_\phi$$

By similar reasoning, we know $\Gamma^{(2)} \propto \sigma^4\phi_0^4$ , so that carrying dimension of mass, $\Gamma^{(2)}$ scales like

$$\Gamma^{(2)} \sim \sigma^4\phi_0^4 m_\phi$$

Thus higher order processes become comparable when

$$\frac{\Gamma^{(2)}}{\Gamma^{(1)}} \sim \sigma^2\phi_0^2 \gtrsim 1$$

Equivalently, using $n_\phi = m_\phi\phi_0^2$ again, higher order processes become significant when

$$\phi_0^2 >> 1/\sigma^2 \qquad \Longleftrightarrow \qquad n_\phi >> m_\phi/\sigma^2 \tag{39}$$

Again, this is corroborated by the fact that for $\Gamma^{(1)} \sim \sigma^2\phi_0^2 m_\phi$, the condition $\Gamma >> m_\phi$ is precisely (39).

## 4.3 Conclusion

We have shown that for some simple interactions, e.g. $\mathcal{L}_{int} = -g\phi\chi^3$ and $\mathcal{L}_{int} = -\sigma\phi\chi\overline{\psi}\psi$, the particle approximation is sufficient to model the decay of coherent scalar field oscillations in the limit of weak interactions and fast oscillations - i.e. in the Born-Oppenheimer approximation $\Gamma << m_\phi$. These conditions arises physically, for example, when $\phi$ is a modulus field coupled gravitationally (weakly) to Standard Model fields. However, when the strength of the interactions become large (as is the case, for example, when $\phi$ is a super-partner of quarks or leptons), the coherence of the field becomes important, and higher order processes contribute significantly to the energy transfer rate. In this case, the Born-Oppenheimer approximation is violated, as $\Gamma$ becomes large. When this approximation breaks down, our justification for averaging over $\phi(t)$ oscillations to obtain a quantum field $\phi$, and for using standard S-matrix techniques becomes invalid, and time-dependent methods become necessary.

Finally, it is important to note that the results obtained thus far are limited to fields interacting in a flat Minkowski universe. In order to establish a closer connection with inflationary cosmology, these computational approaches must be examined in an expanding universe. In addition, coherently oscillating fields in nature are likely to contain higher frequency modes. More realistically, in the classical background treatment, we may wish to consider coherently oscillating scalar fields of the form $\phi(t) = \varphi_0 + \sum_{n=1}^{\infty} \phi_0 \cos(n m_\phi t)$.

# Appendices

# A Decay in the Particle Approach

## A.1 The $\phi \to \chi + \chi + \chi$ Decay

In this section we give the details for the calculation of the $\phi \to \chi + \chi + \chi$ decay rate via the $g\phi\chi^3$, outlined in section 2.1 Consider the general case of a scalar field $\phi$ of mass $m_\phi$ and four-momentum $p_0$ decaying in to three (arbitrary) distinguishable particles of masses, $m_1$, $m_2$, and $m_3$, and four-momenta $p_1$, $p_2$, and $p_3$, respectively - as depicted in figure (1). The general expression for the decay rate differential is given by Fermi's golden rule as

$$d\Gamma = \frac{1}{2m_\phi}\left(\prod_{i=1}^{3}\frac{d^3p_i}{(2\pi)^3 2E_i}\right)|\mathcal{M}|^2(2\pi)^4\delta^{(4)}(p_0 - \sum_{i=1}^{3}p_i) \tag{40}$$

Integrating over $d^3p_1$ and making use of the momentum delta-function, we obtain

$$\begin{aligned} d\Gamma &= \frac{1}{2m_\phi}\frac{d^3p_2\, d^3p_3}{(2\pi)^5 8E_1E_2E_3}|\mathcal{M}|^2\delta(E_0 - E_1 - E_2 - E_3) \\ &= \frac{|\mathbf{p}_2||\mathbf{p}_3|dE_2dE_3d\Omega_2d\Omega_3}{16m_\phi(2\pi)^5E_1}|\mathcal{M}|^2\delta(E_0 - E_1 - E_2 - E_3) \end{aligned} \tag{41}$$

where I have introduced $d^3p_i = |\mathbf{p}_i|^2dp_id\Omega_i$, and utilized $|\mathbf{p}_i|dp_i = E_idE_i$. Denote the $\mathbf{p}_2$ angular variables by $d\Omega_2 = d\phi_2 d\cos\theta_2$, and let $d\Omega_3 = d\phi_3 d\cos\theta_3$ denote the angles of $|\mathbf{p}_3|$ as measure *with respect to the momentum* $p_2$. That is, $\mathbf{p}_2 \cdot \mathbf{p}_3 = |\mathbf{p}_2||\mathbf{p}_3|\cos\theta_3$. We may then intergrate over $d\Omega_2$, yielding a factor of $2(2\pi)^2$:

$$d\Gamma = |\mathcal{M}|^2\frac{|\mathbf{p}_2||\mathbf{p}_3|}{8m_\phi(2\pi)^3E_1}dE_2dE_3d\cos\theta_3\delta(E_0 - E_1 - E_2 - E_3) \tag{42}$$

Now, note that $p_1^2 = (p_0 - p_2 - p_3)^2$ so that in the $m_\phi$ rest frame we have

$$\begin{aligned} E_1^2 &= |\mathbf{p}_1|^2 + m_\phi^2 + m_2^2 + m_3^2 - 2m_\phi E_2 - 2m_\phi E_3 + 2E_2E_3 - 2|\mathbf{p}_2||\mathbf{p}_3|\cos\theta_3 \\ &\equiv \xi - 2|\mathbf{p}_2||\mathbf{p}_3|\cos\theta_3 \end{aligned} \tag{43}$$

and therefore

$$E_1 = \sqrt{\xi - 2|\mathbf{p}_2||\mathbf{p}_3|\cos\theta_3} \tag{44}$$

Recall the general identity $\delta(f(x)) = \delta(x - x_0)\frac{1}{|\frac{df}{dx}(x_0)|}$ where $x_0$ denotes a zero of $f(x)$. We utilize this to obtain

$$\delta(E_0 - E_1 - E_2 - E_3) = \delta(E_0 - E_2 - E_3 - \sqrt{\xi - 2|\mathbf{p}_2||\mathbf{p}_3|\cos\theta_3}) \equiv \delta(u(\cos\theta_3))$$

$$= \delta(\eta - \cos\theta_3)\frac{1}{|\frac{du}{dcos\theta_3}|} \tag{45}$$

where $\eta$ is obtained by solving $u(\cos\theta_3) = 0$ for $\cos\theta_3$. Then, noting that

$$\frac{du}{d\cos\theta_3} = \frac{|\mathbf{p}_2||\mathbf{p}_3|}{\sqrt{\xi - 2|\mathbf{p}_2||\mathbf{p}_3|\cos\theta_3}} = \frac{|\mathbf{p}_2||\mathbf{p}_3|}{E_1} \tag{46}$$

we may integrate over $d\cos\theta_3$ in eq. (42) to obtain the final expression

$$d\Gamma = |\mathcal{M}|^2 \frac{|\mathbf{p}_2||\mathbf{p}_3|}{8m_\phi(2\pi)^3 E_1} dE_2 dE_3 d\cos\theta_3 \frac{E_1}{|\mathbf{p}_2||\mathbf{p}_3|}\delta(\eta - \cos\theta_3)$$

$$= \frac{|\mathcal{M}|^2}{8m_\phi(2\pi)^3} dE_2 dE_3 \tag{47}$$

### A.1.1 Dalitz Coordinates

Define the invariant masses $m_{ij}$ by $m_{ij}^2 = (p_1 + p_2)^2$ for $i$,$j = 1, 2, 3$. It is easy to show that for $k \neq i, j$ we have by four-momentum conservation

$$m_{ij}^2 = (p_0 - p_k)^2 = m_\phi^2 + m_k^2 - 2m_\phi E_k \tag{48}$$

so that

$$dE_k = -\frac{dm_{ij}^2}{2m_\phi} \tag{49}$$

Thus, replacing $dE_2$ and $dE_3$ in eq.47 we obtain

$$d\Gamma = \frac{|\mathcal{M}|^2}{32m_\phi^3(2\pi)^3}dm_{12}^2 dm_{13}^2 \tag{50}$$

To perform the integral, we must identify the bounds on $m_{12}$ and $m_{13}^2$. First notice that

$$m_{13}^2 = (p_1 + p_3)^2 = m_1^2 + m_3^2 + 2E_1E_3 - 2\mathbf{p}_1 \cdot \mathbf{p}_3 \tag{51}$$

so that $m_{13}^2$ is maximum when $p_1$ and $p_3$ are anti-parallel, and minimum when they are parallel. Thus we have

$$\begin{aligned} m_{13-max}^2 &= (p_1 + p_3)^2 = (E_1 + E_3)^2 - (|\mathbf{p}_1| - |\mathbf{p}_3|)^2 \\ &= (E_1 + E_3)^2 - \left(\sqrt{E_1^2 - m_1^2} - \sqrt{E_3^2 - m_3^2}\right)^2 \end{aligned} \tag{52}$$

and

$$\begin{aligned} m_{13-min}^2 &= (p_1 + p_3)^2 = (E_1 + E_3)^2 - (|\mathbf{p}_1| + |\mathbf{p}_3|)^2 \\ &= (E_1 + E_3)^2 - \left(\sqrt{E_1^2 - m_1^2} + \sqrt{E_3^2 - m_3^2}\right)^2 \end{aligned} \tag{53}$$

Now, given that the above quantities are frame-invariant, we may boost to the $m_{12}$ rest frame and use the relations

$$p_{12} \cdot p_1 = m_{12}E_1' = m_1^2 + E_1'E_2' - \mathbf{p}_1 \cdot \mathbf{p}_2 \tag{54}$$

and

$$m_{12}^2 = m_1^2 + m_2^2 + 2E_1'E_2' - 2\mathbf{p}_1 \cdot \mathbf{p}_2 \tag{55}$$

to find

$$E_1' = \frac{m_{12}^2 + m_1^2 - m_2^2}{2m_{12}} \tag{56}$$

and from $p_0^2 = m_\phi^2 = (p_1 + p_2 + p_3)^2 = m_{12}^2 + m_3^2 + 2E_3' m_{12}$ we find

$$E_3' = \frac{m_\phi^2 - m_3^2 - m_{12}^2}{2m_{12}} \tag{57}$$

Therfore, we see that for fixed $m_{12}^2$, $m_{13}^2$ is bounded above and below (respectively) by the curves

$$m_{13-max}^2 = \frac{1}{4m_{12}^2}\Bigg[(m_\phi + m_1^2 - m_2^2 - m_3^2) - \Bigg(\sqrt{-4m_1^2m_{12}^2 + (m_1^2 + m_{12}^2 - m_2^2)^2} - \sqrt{-4m_3^2m_{12}^2 + (-m_3^2 + m_{12}^2 + m_3^2)^2}\Bigg)^2\Bigg] \tag{58}$$

$$m_{13-min}^2 = \frac{1}{4m_{12}^2}\Bigg[(m_\phi + m_1^2 - m_2^2 - m_3^2) - \Bigg(\sqrt{-4m_1^2m_{12}^2 + (m_1^2 + m_{12}^2 - m_2^2)^2} + \sqrt{-4m_3^2m_{12}^2 + (-m_3^2 + m_{12}^2 + m_3^2)^2}\Bigg)^2\Bigg] \tag{59}$$

In the special case where the decay products are massless ($m_1 = m_2 = m_3 = 0$), the bounds eqs. (58, 59) reduce to

$$m_{13-max}^2 = m_\phi^2 - m_{12}^2 \tag{60}$$

$$m_{13-min}^2 = 0 \tag{61}$$

## A.2 The $\phi \to \chi + \overline{\psi} + \psi$ Decay

We consider the first order process depicted in figure 4 The first order S-matrix amplitude for this process is given by $i\mathcal{M} = \sigma v^r(p_3)\bar{u}^s(p_2)$ so that

$$
\begin{aligned}
|\mathcal{M}|^2 &= \sigma^2(v^r(p_3)\bar{u}^s(p_2))(v^r(p_3)\bar{u}^s(p_2)) \\
&= \sigma^2(v^r(p_3)\bar{u}^s(p_2))(u^s(p_2)\bar{v}^r(p_3)) \\
&= \sigma^2\sum_{i,j}(v^r(p_3)_i\bar{u}^s(p_2)_i)(u^s(p_2)_j\bar{v}^r(p_3)_j)
\end{aligned}
\tag{62}
$$

Summing over the spins of the final states gives

$$
\begin{aligned}
\sum_{r,s}|\mathcal{M}|^2 &= \sigma^2\sum_{r,s}\sum_{i,j}(v^r(p_3)_i\bar{u}^s(p_2)_i)(u^s(p_2)_j\bar{v}^r(p_3)_j) \\
&= \sigma^2\sum_{i,j}(\not{p}_3 - m_3)_{ij}(\not{p}_2 + m_2)_{ji} = \mathrm{tr}\{(\not{p}_3 - m_3)(\not{p}_2 + m_2)\} \\
&= \sigma^2\,\mathrm{tr}\{\not{p}_3\not{p}_2 - m_3\not{p}_2 + m_2\not{p}_3 - m_2m_3\} \\
&= \sigma^2\,\mathrm{tr}\{\not{p}_3\not{p}_2 - m_2m_3\} = \sigma^2(\mathrm{tr}\{\not{p}_3\not{p}_2\} - \mathrm{tr}\{m_2m_3\}) \\
&= \sigma^2(\mathrm{tr}\{\not{p}_3\not{p}_2\} - 4m_2m_3)
\end{aligned}
\tag{63}
$$

Now, denoting the Minkowski metric by $\eta_{\mu\nu}$, we have

$$
\begin{aligned}
\mathrm{tr}\{(p_3\cdot\gamma)(p_2\cdot\gamma)\} &= \sigma^2\,\mathrm{tr}\{\eta_{\nu\mu}p_3^\mu\gamma^\nu\eta_{\mu'\nu'}p_2^{\mu'}\gamma^{\nu'}\} \\
&= \eta_{\mu\nu}\eta_{\mu'\nu'}p_3^\mu p_2^{\mu'}\,\mathrm{tr}\{\gamma^\nu\gamma^{\nu'}\} \\
&= \eta_{\mu\nu}\eta_{\mu'\nu'}p_3^\mu p_2^{\mu'}(4\eta^{\nu\nu'}) \\
&= 4p_2\cdot p_3
\end{aligned}
\tag{64}
$$

so that

$$
\sum_{r,s}|\mathcal{M}|^2 = 4\sigma^2(p_2\cdot p_3 - m_2m_3) \tag{65}
$$

Finally, using Fermi's golden (40) with a sum over spins, we obtain

$$d\Gamma = \frac{1}{2m_\phi}\left(\prod_{i=1}^{3}\frac{d^3p_i}{(2\pi)^3 2E_i}\right)\sum_{r,s}|\mathcal{M}|^2(2\pi)^4\delta^{(4)}(p_0 - \sum_{i=1}^{3}p_i) \tag{66}$$

$$\begin{aligned} d\Gamma &= \frac{\sum_{r,s}|\mathcal{M}|^2}{8m_\phi(2\pi)^3}dE_2dE_3 = \sigma^2\frac{p_2\cdot p_3 - m_2m_3}{2m_\phi(2\pi)^3}dE_2dE_3 \\ &= \sigma^2\frac{E_2E_3 - \mathbf{p}_2\cdot\mathbf{p}_3 - m_2m_3}{2m_\phi(2\pi)^3}dE_1dE_2 \\ &= \sigma^2\frac{E_2E_3 - |\mathbf{p}_2||\mathbf{p}_3|\cos\theta_3 - m_2m_3}{2m_\phi(2\pi)^3}dE_1dE_2 \end{aligned} \tag{67}$$

Now, recall from the delta function in (45) that we have $\cos\theta_3 = \eta$ where $\eta$ is obtained from solving $u(\cos\theta_3) = 0$. In particular, $\eta$ is obtained through

$$\begin{aligned} |\mathbf{p}_2||\mathbf{p}_3|\eta &= -\frac{1}{2}\Big[(E_2 + E_3 - E_0)^2 - \xi\Big] \\ &= -\frac{1}{2}\Big[(E_2 + E_3 - E_0)^2 - (|\mathbf{p}_1|^2 + m_\phi^2 + m_2^2 + m_3^2 - 2m_\phi E_2 - 2m_\phi E_3 + 2E_2E_3)\Big] \\ &= \frac{1}{2}(|\mathbf{p}_1|^2 + m_2^2 + m_3^2 - E_2^2 - E_3^2) \\ &= \frac{1}{2}(-m_1^2 + m_2^2 + m_3^2 + E_1^2 - E_2^2 - E_3^2) \end{aligned} \tag{68}$$

We can simplify part of this expression as

$$E_2E_3 - \frac{1}{2}(-m_1^2 + m_2^2 + m_3^2 + E_1^2 - E_2^2 - E_3^2) - m_2m_3 = \frac{1}{2}\Big[(E_2 + E_3)^2 - E_1^2 - (m_2 + m_3)^2 + m_1^2\Big]$$

giving us

$$d\Gamma = \sigma^2\frac{(E_2 + E_3)^2 - E_1^2 - (m_2 + m_3)^2 + m_1^2}{4m_\phi(2\pi)^3}dE_2dE_3 \tag{69}$$

Finally, to convert to the Dalitz coordinates, first note that by energy conservation, we have $(E_2 + E_3)^2 - E_1^2 = -m_\phi^2 + 2m_\phi E_2 + 2m_\phi E_3$, while $m_{12}^2 = m_\phi^2 + m_2^2 - 2m_\phi E_3$ and $m_{13}^2 = m_\phi^2 + m_3^2 - 2m_\phi E_2$ so that

$$d\Gamma = \sigma^2 \frac{m_\phi^2 + m_1^2 - 2m_2m_3 - (m_{12}^2 + m_{13}^2)}{16m_\phi^3(2\pi)^3} dm_{12}^2 dm_{13}^2 \tag{70}$$

# B The Classical Background Approach

## B.1 The $\phi\chi^3$ Interaction

In this section we examine the decay of a coherently oscillating background field $\phi(x) = \phi(\mathbf{x}, t) = \phi_0 \cos(m_\phi t)$ into three scalar $\chi$ particles via the interaction Hamiltonian

$$H_{\text{int}} = g\phi\chi^3$$

In particular, we evaluate the vacuum decay rate via (40), where the amplitude for the production of $\chi$ particles of momenta $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, is given by

$$\mathcal{M} = \langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3| \exp\left\{\int_{-T}^{T} dt \int d^3\mathbf{x}\, H_{int}(t)\right\}|0\rangle \tag{71}$$

$$= \langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3| \int_{-T}^{T} dt \int d^3\mathbf{x}\, g\phi(t)\chi^3(x)|0\rangle + \cdots \tag{72}$$

$$= ig\phi_0 \int_{-T}^{T} dt \int d^3\mathbf{x} \langle \mathbf{p_1 p_2 p_3}| \cos(m_\phi t)\chi^3(x)|0\rangle + \cdots \tag{73}$$

$$\tag{74}$$

We expand the fields $\chi$ using

$$\chi(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2E_\mathbf{k}}} a_\mathbf{k}^\dagger e^{ik\cdot x} + a_\mathbf{k} e^{-ik\cdot x} \tag{75}$$

where $a_\mathbf{p}$ and $a_{\mathbf{p}'}^\dagger$ obey the commutation relations $[a_\mathbf{p}, a_{\mathbf{p}'}^\dagger] = (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{p}')$

In the limit of small $g$, we expand the integral (71) and remove terms which annihilate the vacuum to obtain

$$
\begin{aligned}
\mathcal{M} &= ig\phi_0 \int_{-T}^{T} dt \int d^3\mathbf{x}\ \cos(m_\phi t) \int \frac{d^3\mathbf{k}_1\ d^3\mathbf{k}_2\ d\mathbf{k}_3}{(2\pi)^9\sqrt{2E_1 2E_2 2E_3}} \langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 | a^\dagger_{\mathbf{k}_1} a^\dagger_{\mathbf{k}_2} a^\dagger_{\mathbf{k}_3} |0\rangle e^{i(k_1+k_2+k_3)\cdot x} \\
&= ig\phi_0 \int_{-T}^{T} dt \int d^3\mathbf{x}\ \cos(m_\phi t) \int \frac{d^3\mathbf{k}_1\ d^3\mathbf{k}_2\ d\mathbf{k}_3}{(2\pi)^9\sqrt{2E_1 2E_2 2E_3}} e^{i(k_1+k_2+k_3)\cdot x} \\
&\qquad \times \sqrt{2E_1 2E_2 2E_3} \langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 | \mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle \\
&= ig\phi_0 \int_{-T}^{T} dt \int d^3\mathbf{x}\ \cos(m_\phi t) \int\ d^3\mathbf{k}_1\ d^3\mathbf{k}_2\ d^3\mathbf{k}_3 e^{i(k_1+k_2+k_3)\cdot x} \\
&\qquad \times \delta^{(3)}(\mathbf{k}_1 - \mathbf{p}_1)\delta^{(3)}(\mathbf{k}_2 - \mathbf{p}_2)\delta^{(3)}(\mathbf{k}_3 - \mathbf{p}_3) \\
&= ig\phi_0 \int_{-T}^{T} dt \int d^3\mathbf{x}\ \cos(m_\phi t) e^{i(p_1+p_2+p_3)\cdot x} \\
&= ig\phi_0 \int_{-T}^{T} dt \int d^3\mathbf{x}\ \left(\frac{e^{-im_\phi t} + e^{im_\phi t}}{2}\right) e^{i(p_1+p_2+p_3)\cdot x} \\
&= ig\frac{\phi_0}{2} \int_{-T}^{T} \int d^3\mathbf{x} \Big( \exp\left\{(E_1+E_2+E_3-m_\phi)t - (\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\cdot\mathbf{x}\right\} \\
&\qquad + \exp\{(E_1+E_2+E_3+m_\phi)t - (\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\cdot\mathbf{x}\}\Big) \\
&= ig\phi_0 \left[\frac{\sin\left((E_1+E_2+E_3-m_\phi)T\right)}{(E_1+E_2+E_3-m_\phi)} + \frac{\sin\left((E_1+E_2+E_3+m_\phi)T\right)}{(E_1+E_2+E_3+m_\phi)}\right] \\
&\qquad \times (2\pi)^3\delta^{(3)}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)
\end{aligned} \tag{76}
$$

Taking the square of the amplitude, we obtain

$$
\begin{aligned}
|\mathcal{M}|^2 = g^2\phi_0^2 \Bigg[ &\frac{\sin^2\left((E_1+E_2+E_3-m_\phi)T\right)}{(E_1+E_2+E_3-m_\phi)^2} + \frac{\sin^2\left((E_1+E_2+E_3+m_\phi)T\right)}{(E_1+E_2+E_3+m_\phi)^2} \\
&+ 2\frac{\sin\left((E_1+E_2+E_3-m_\phi)T\right)\sin\left((E_1+E_2+E_3+m_\phi)T\right)}{(E_1+E_2+E_3)^2 - m_\phi^2}\Bigg] \\
&\times V \times (2\pi)^3\delta^{(3)}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)
\end{aligned} \tag{77}
$$

where I have used the fact that under the integral over $\prod_{i=1}^{3} d^3\mathbf{p}_i$,we can write

$$
\left(\delta^{(2)}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\right)^2 = \delta^{(3)}(\mathbf{0})\delta^{(3)}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) = V \times \delta^{(3)}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)
$$

where $V$ is the (possibly infinite) volume of the phase space.

## B.2 The $\phi\chi\overline{\psi}\psi$ Interaction

In this section, we give the details for computing the amplitude for vacuum excitation resulting from the time-dependent interaction

$$H_{int} = \sigma\phi(t)\chi\overline{\psi}\psi$$

with the coherently oscillating scalar field $\phi(t) = \phi_0 \cos(m_\phi t)$. In particular, we evaluate first order amplitude for the production of $\chi$, $\psi$, and $\overline{\psi}$ particles of momenta $\mathbf{p}_1$, $\mathbf{p}_2$,and $\mathbf{p}_3$. To first order in $\sigma$, the amplitude is given by

$$\begin{aligned}
\mathcal{M} &= \langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3| \exp\left\{\int_{-T}^{T} dt \int d^3\mathbf{x}\, H_{int}(t)\right\}|0\rangle \\
&= \langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3| \int_{-T}^{T} dt \int d^3\mathbf{x}\, \sigma\phi(t)\chi(x)\overline{\psi}(x)\psi(x)|0\rangle + \cdots \\
&= i\sigma\phi_0 \int_{-T}^{T} dt \int d^3\mathbf{x} \langle \mathbf{p_1p_2p_3}| \cos(m_\phi t)\chi(x)\overline{\psi}(x)\psi(x)|0\rangle + \cdots
\end{aligned} \tag{78}$$

We expand the fields $\chi$, $\psi$, and $\overline{\psi}$ using

$$\chi(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2E_\mathbf{k}}} a^\dagger_\mathbf{k} e^{ik\cdot x} + a_\mathbf{k} e^{-ik\cdot x} \tag{79}$$

$$\psi = \int \frac{d^3k}{(2\pi)^3\sqrt{2E_\mathbf{k}}} \sum_s \left(a^s_\mathbf{k} u^s(k) e^{-ik\cdot x} + b^{s\dagger}_\mathbf{k} v^s(k) e^{ik\cdot x}\right) \tag{80}$$

$$\overline{\psi} = \int \frac{d^3k}{(2\pi)^3\sqrt{2E_\mathbf{k}}} \sum_s \left(b^s_\mathbf{k} \overline{v}^s(k) e^{-ik\cdot x} + a^{s\dagger}_\mathbf{k} \overline{u}^s(k) e^{ik\cdot x}\right) \tag{81}$$

where $[a_\mathbf{p}, a^\dagger_{\mathbf{p}'}] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')$, and $\{a^r_\mathbf{p}, a^{s\dagger}_{\mathbf{p}'}\} = \{b^r_\mathbf{p}, b^{s\dagger}_{\mathbf{p}'}\} = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{rs}$.

Substituting these field expansions into the expression 78, using the commutation relations to commute creation/annihilation operators, and removing terms which annihilate the vacuum, we find that the amplitude becomes

$$\mathcal{M} = i\sigma\phi_0 \int_{-T}^{T} dt \int d^3\mathbf{x}\ \cos(m_\phi t) \int \frac{d^3\mathbf{k}_1\, d^3\mathbf{k}_2\, d\mathbf{k}_3}{(2\pi)^9\sqrt{2E_1 2E_2 2E_3}} \tag{82}$$

$$\times \langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 | v^r(k_3)\overline{u}^s(k_2) a^\dagger_{\mathbf{k}_1} a^s_{\mathbf{k}_2} b^r_{\mathbf{k}_3} |0\rangle \tag{83}$$

$$= i\sigma\phi_0 \int_{-T}^{T} dt \int d^3\mathbf{x}\ \cos(m_\phi t) \int \frac{d^3\mathbf{k}_1\, d^3\mathbf{k}_2\, d\mathbf{k}_3}{(2\pi)^9\sqrt{2E_1 2E_2 2E_3}} e^{i(k_1+k_2+k_3)\cdot x}$$
$$\times \sqrt{2E_1 2E_2 2E_3}\ v^r(k_3)\overline{u}^s(k_2)\langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3 | \mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\rangle$$
$$= i\sigma\phi_0 \int_{-T}^{T} dt \int d^3\mathbf{x}\ \cos(m_\phi t) \int\ d^3\mathbf{k}_1\, d^3\mathbf{k}_2\, d^3\mathbf{k}_3 e^{i(k_1+k_2+k_3)\cdot x}$$
$$\times\ v^r(k_3)\overline{u}^s(k_2)\delta^{(3)}(\mathbf{k}_1 - \mathbf{p}_1)\delta^{(3)}(\mathbf{k}_2 - \mathbf{p}_2)\delta^{(3)}(\mathbf{k}_3 - \mathbf{p}_3)$$
$$= i\sigma\phi_0 \int_{-T}^{T} dt \int d^3\mathbf{x}\ \cos(m_\phi t) v^r(p_3)\overline{u}^s(p_2) e^{i(p_1+p_2+p_3)\cdot x}$$
$$= i\sigma\phi_0\ v^r(p_3)\overline{u}^s(p_2) \int_{-T}^{T} dt \int d^3\mathbf{x}\ \left(\frac{e^{-im_\phi t} + e^{im_\phi t}}{2}\right) e^{i(p_1+p_2+p_3)\cdot x}$$
$$= i\sigma\frac{\phi_0}{2}\ v^r(p_3)\overline{u}^s(p_2) \int_{-T}^{T} \int d^3\mathbf{x} \Big( \exp\{(E_1+E_2+E_3-m_\phi)t - (\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\cdot\mathbf{x}\}$$
$$+ \exp\{(E_1+E_2+E_3+m_\phi)t - (\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\cdot\mathbf{x}\}\Big)$$
$$= i\sigma\frac{\phi_0}{2}\ v^r(p_3)\overline{u}^s(p_2)\Bigg[\frac{\sin\left((E_1+E_2+E_3-m_\phi)T\right)}{(E_1+E_2+E_3-m_\phi)}$$
$$+ \frac{\sin\left((E_1+E_2+E_3+m_\phi)T\right)}{(E_1+E_2+E_3+m_\phi)}\Bigg] \times (2\pi)^3\delta^{(3)}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) \tag{84}$$

Then, squaring the amplitude and summing over spin-states, we find (using the results from the spin-summation in A.2) that

$$\sum_{r,s}|\mathcal{M}|^2 = 4\sigma^2\phi_0^2(p_2\cdot p_3 - m_2 m_3)\Bigg[\frac{\sin^2\left((E_1+E_2+E_3-m_\phi)T\right)}{(E_1+E_2+E_3-m_\phi)^2} \tag{85}$$
$$+ \frac{\sin^2\left((E_1+E_2+E_3+m_\phi)T\right)}{(E_1+E_2+E_3+m_\phi)^2}$$
$$+ 2\frac{\sin\left((E_1+E_2+E_3-m_\phi)T\right)\sin\left((E_1+E_2+E_3+m_\phi)T\right)}{(E_1+E_2+E_3)^2 - m_\phi^2}\Bigg]$$
$$\times V \times (2\pi)^3\delta^{(3)}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) \tag{86}$$

where I have again used the fact that under the integral over $\prod_{i=1}^{3} d^3\mathbf{p}_i$,we can write

$$\left(\delta^{(2)}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\right)^2 = \delta^{(3)}(\mathbf{0})\delta^{(3)}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) = V \times \delta^{(3)}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)$$

where $V$ is the (possibly infinite) volume of the phase space.

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